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Finite point transformations and linearisation of $\ddot{x} = f(t,x)$

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Abstract. The usefulness of finite point transformations is emphasised, as a systematic tool for studying the eventual linearisation of ordinary second-order differential equations.

Recently, the question of linearisation of non-linear differential equations has been considered as an interesting starting point for studying their symmetry properties (Sarlet *et al* 1987). Of course, by linearisation of a differential equation one means that the equation becomes linear after performing a suitable non-singular point transformation. The interest of linearisation stems from the fact that *all* ordinary linear second-order differential equations have SL(3,R) as the maximal point symmetry group. Indeed, in the context of Lie's theory of extended groups (Lie 1967) it is well known that all such equations exhibit eight symmetries whose generators obey the sl(3, R) Lie algebra (Wulfman and Wybourne 1976, Leach 1980, Aguirre and Krause 1984, 1988b). It is further known that, as a consequence of Lie's counting theorem, second-order non-linear differential equations can have *at most* eight point symmetries (Anderson and Davison 1974, Ovsiannikov 1978, Aguirre and Krause 1985, Leach 1985), while some of them have no symmetry at all. As a matter of fact, there are well known instances of non-linear equations which exhibit eight point symmetries, whose generators again satisfy the sl(3, R) commutation relations (Sarlet *et al* 1987). Therefore, since the commutation relations of symmetry generators are invariant under arbitrary point transformations, it follows that linearisation is a necessary and sufficient condition for a second-order differential equation to admit SL(3, R) as the maximal point symmetry group.

All these interesting results (as well as many others that figure in the literature (cf Bluman and Cole 1974, Ovsiannikov 1978)) have been established by means of Lie's 'infinitesimal transformation' approach to extended point symmetry group actions, which is a powerful standard technique indeed. Notwithstanding this fact, in two recent papers (Aguirre and Krause 1987, 1988a) we have adopted a different, albeit rather natural, approach to the issue of symmetries of linear differential equations. In fact, we use *finite point transformations* (instead of Lie's infinitesimal transformations) to uncover the symmetry group. In this fashion, a unified and exhaustive treatment arises for studying the similarity properties of such equations, since we calculate the finite diffeomorphisms (with their eight parameters included), which correspond to non-linear realisations of SL(3, R) acting as the maximal group of symmetry transformations for any given second-order linear differential equation in one dimension. It turns out that these diffeomorphisms are given by conjugations of the form $F^{-1}\mathrm{P}_2F$,

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of the projective group P_2 of the plane and a non-singular parameter-free point transformation F (Aguirre and Krause 1988a). Of course, the first-order infinitesimal versions of these diffeomorphisms yield the realisations of the eight generators and, hence, the relevant realisation of the sl(3, R) Lie algebra follows, as usual (Aguirre and Krause 1988b).

The purpose of the present paper is to illustrate some of the huge possibilities of the finite transformation approach, while examining briefly the problem of linearisation of a second-order differential equation under the scope of this new perspective. Perhaps none of our results is really new, though we have not seen the explicit formulae in the current literature. As a matter of fact, our aim here is merely instrumental, since the emphasis of this paper lies in showing some of the advantages of the *finite transformation method* for similarity analysis.

To this end, let us start from the assumption that the differential equation under investigation is of the general non-linear form

$$\ddot{x}=\sum_n f_n(t, x)\dot{x}^n. \tag{1}$$

Since we are interested in general conditions for linearisation, we search those finite point transformations (if any),

$$\begin{aligned} t' &= T(t, x) \\ x' &= S(t, x) \end{aligned} \tag{2}$$

with non-vanishing Jacobian

$$J(t, x) = T_t S_x - T_x S_t \tag{3}$$

which reduce equation (1) to a linear second-order differential equation. Thus we demand

$$\ddot{x}=\sum_n f_n(t, x)\dot{x}^n \Rightarrow \ddot{x}'=0 \tag{4}$$

(where $\ddot{x}'=\mathrm{d}^2x'/\mathrm{d}t'^2$), which should hold upon the finite transformation of variables (2), by hypothesis. Namely, equation (1) is linearisable if it is linearisable through a point transformation like (1). As a matter of fact, if a second-order differential equation is linearisable, then one can demand *full linear reduction* of the equation, as in (4), without loss of generality (cf, for instance, Aguirre and Krause 1988a).

Now, the first two extended transformations of (2) (i.e. $\dot{x}\to\dot{x}'$ and $\ddot{x}\to\ddot{x}'$) are given by

$$\dot{x}'=(T_t+\dot{x}T_x)^{-1}(S_t+\dot{x}S_x) \tag{5}$$

$$\ddot{x}'=(T_t+\dot{x}T_x)^{-3}[(T_t+\dot{x}T_x)(\mathrm{d}/\mathrm{d}t)(S_t+\dot{x}S_x)-(S_t+\dot{x}S_x)(\mathrm{d}/\mathrm{d}t)(T_t+\dot{x}T_x)]. \tag{6}$$

In consequence, a necessary and sufficient condition for a second-order differential equation to be linearisable is provided by the following identity:

$$\begin{aligned} &T_tS_{tt}-T_{tt}S_t+(2T_tS_{tx}+T_xS_{tt}-2S_tT_{tx}-S_xT_{tt})\dot{x} \\ &\qquad +(2T_xS_{xt}+T_tS_{xx}-2S_xT_{xt}-S_tT_{xx})\dot{x}^2+(T_xS_{xx}-T_{xx}S_x)\dot{x}^3 \\ &\qquad +(T_tS_x-T_xS_t)\sum_n f_n\dot{x}^n\equiv 0. \end{aligned} \tag{7}$$

This relation must hold for all $\dot{x}$, and therefore, after separating the coefficients of the different powers of $\dot{x}$, one obtains a system of coupled differential equations for T and S, in terms of the given f_n. Indeed, one gets

$$T_t S_{tt} - T_{tt} S_t + J f_0 = 0 \tag{8}$$

$$2T_t S_{tx} + T_x S_{tt} - 2T_{tx} S_t - T_{tt} S_x + J f_1 = 0 \tag{9}$$

$$2T_x S_{xt} + T_t S_{xx} - 2T_{xt} S_x - T_{xx} S_t + J f_2 = 0 \tag{10}$$

$$T_x S_{xx} - T_{xx} S_x + J f_3 = 0 \tag{11}$$

and, moreover,

$$J f_n = 0 \qquad \text{for } n = 4, 5, \ldots, \text{ and } n = -1, -2, \ldots. \tag{12}$$

Thus we see that in this approach one recovers, almost immediately (that is, at the very beginning of the analysis), a well known feature concerning linearisability of ordinary second-order differential equations (Aguirre and Krause 1985, Sarlet *et al* 1987), i.e. all linearisable second-order differential equations must be, at most, polynomials of the third degree in $\dot{x}$. That is, a *necessary* condition for linearisation is that the differential equation be of the form:

$$\ddot{x} = f_0(t, x) + f_1(t, x)\dot{x} + f_2(t, x)\dot{x}^2 + f_3(t, x)\dot{x}^3. \tag{13}$$

Of course, one can also show this important theorem by means of the standard tools offered by Lie's infinitesimal method; the proof, however, turns out to be much more involved (cf, for instance, Aguirre and Krause (1985)).

Clearly, condition (13) is *not sufficient* for linearisation. Nevertheless, the finite transformation approach to this issue also yields a constructive procedure in order to try to linearise a given equation of this form. Plainly so, since a necessary and sufficient condition for an equation of the form (13) to be linearisable is that the system of differential equations (8)-(11) has solutions for T and S (with $J \neq 0$).

As a simple example of the power of the finite transformation approach to similarity analysis, in this paper we will also discuss the linearisation problem of differential equations of a rather special type, i.e. henceforth we consider

$$\ddot{x} = f(t, x) \tag{14}$$

where f is a given function of t and x. These are the simplest kind of differential equations of the form (13), and they deserve some interest by themselves. Moreover, let us also recall that every equation of the form $\ddot{x} = f(t, x) + g(t)\dot{x}$ may be cast into the form stated in (14) by means of the familiar transformation

$$x \to x \exp\left(\frac{1}{2}\int_0^t \mathrm{d}t'\, g(t')\right)$$

and so it can be linearised if its reduced form (14) is linearisable.

Therefore, according to the previous results, in order to linearise (14) one considers the system of equations (8)-(11), which now becomes

$$T_t S_{tt} - T_{tt} S_t + (T_t S_x - T_x S_t) f(t, x) = 0 \tag{15}$$

$$2T_t S_{tx} + T_x S_{tt} - 2T_{tx} S_t - T_{tt} S_x = 0 \tag{16}$$

$$2T_x S_{tx} + T_t S_{xx} - 2T_{xt} S_x - T_{xx} S_t = 0 \tag{17}$$

$$T_x S_{xx} - T_{xx} S_x = 0. \tag{18}$$

One can formally integrate these equations in a straightforward manner. (For the sake of briefness we omit the details of these calculations.)

Firstly, one assumes $T_x \neq 0$. Then one obtains:

$$S(t, x) = \phi_1(t) T(t, x) + \phi_2(t) \tag{19}$$

$$T(t, x) = [\phi_3(t)x + \phi_4(t)]^{-1} - \dot{\phi}_2(t)/\dot{\phi}_1(t) \tag{20}$$

where the ϕ are functions of t which have to satisfy

$$\phi_2(t) = C_1\phi_1(t) + C_2 \tag{21}$$

$$\phi_3(t) = C_3(\dot{\phi}_1(t))^{1/2} \qquad \text{(i.e. } \dot{\phi}_1(t) \geqslant 0) \tag{22}$$

$$\phi_3(t) f(t, x) = (\ddot{\phi}_1\dot{\phi}_3/\dot{\phi}_1 - \ddot{\phi}_3)x + (\ddot{\phi}_1\dot{\phi}_4/\dot{\phi}_1 - \ddot{\phi}_4) \tag{23}$$

and where the C are arbitrary constants of integration. (Clearly $\phi_3(t) \neq 0$, since $T_x \neq 0$.)

Secondly, if on the other hand one assumes $T_x = 0$, from the modified equations (15)-(17) one gets:

$$S(t, x) = \psi_1(t)x + \psi_2(t) \tag{24}$$

$$\dot{T}(t) = C_0(\psi_1(t))^2 \qquad \text{(i.e. } \psi_1(t) \neq 0) \tag{25}$$

where the ψ have to satisfy

$$[\psi_1(t)]^2 f(t, x) = (2\dot{\psi}_1^2 - \psi_1\ddot{\psi}_1)x + (2\dot{\psi}_1\dot{\psi}_2 - \psi_1\ddot{\psi}_2). \tag{26}$$

Hence, a mere glance at equations (23) and (26) yields the important conclusion: non-linear differential equtions of the form $\ddot{x} = f(t, x)$ are intrinsically non-linear (i.e. they are *not* linearisable). This is a nice result, perhaps well known to some people working in this field. However, as we have already remarked, we have been unable to find it in the literature.

In view of this short paper, and also taking into account our previous work on this subject (Aguirre and Krause 1987, 1988a, b), it seems reasonable to conclude that finite transformations in similarity analysis afford a powerful method, worthy of further research. Work is in progress concerning the use of finite transformation techniques for studying the symmetries of second-order *non-linear* ordinary differential equations in general.

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